Generalized Quadrature Domains

with connections to Hele-Shaw flow

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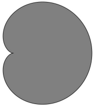

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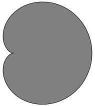

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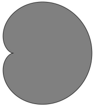

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These are examples of *quadrature identities*.

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Quadrature domains

Definition 1.1 (Quadrature domain)

We call a domain $\Omega \subset \widehat{\mathbb{C}}$ a *quadrature domain* if there exists $h \in \mathrm{Rat}(\Omega)$ s.t.[2]

$$\frac{1}{\pi}\int_{\Omega} f dA = \frac{1}{2\pi i}\oint_{\partial\Omega} f(w)h(w)dw$$

$\forall f \in L^1_a(\Omega)$. This is denoted by $\Omega \in \mathrm{QD}(h)$. (we also assume $\infty \notin \partial\Omega$)

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Residue theorem (Ω bounded) $\rightarrow$ quadrature domain $\iff$ quadrature identity:

$$\frac{1}{2\pi i}\oint_{\partial\Omega} f(w)h(w)dw = \sum_{\text{poles of } h,\ \{p_k\}} \operatorname*{Res}_{w=p_k}(f(w)h(w)) = \sum_{k,j} c_{k,j} f^{(n_j)}(p_k).$$

Unbounded case is similar.

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Can also write $\displaystyle \frac{1}{\pi}\int_{\Omega} f dA = \mu(f)$, where $\displaystyle \mu = \bar{\partial} h = \sum_{k,j} c_{k,j}(-1)^{n_j+1}\delta^{(n_j)}_{p_k}$

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Unbounded QD Example: The deltoid

The complement of the deltoid is an unbounded quadrature domain, $\Omega = \left\{z + \frac{1}{2z^2} : |z| > 1\right\} \in \mathrm{QD}\left(\frac{w^2}{2}\right)$:

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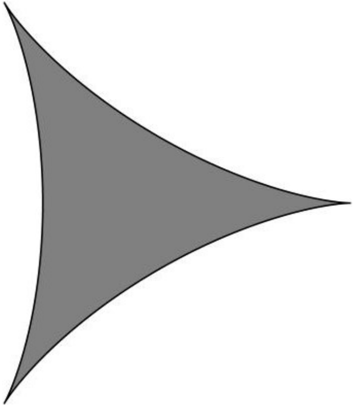

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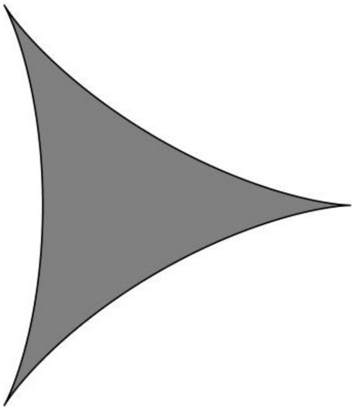

$$\frac{1}{\pi}\int_\Omega f dA = \frac{1}{2\pi i}\oint_{\partial\Omega} f(w)\frac{w^2}{2}dw$$
$$= \frac{1}{2}f_3$$

$$f(w) = f_1 w^{-1} + f_2 w^{-2} + f_3 w^{-3} + \cdots$$

$$(f \in L_a^1 \implies f_1 = f_2 = 0)$$

Potential theoretic interpretation of QDs

Definition 1.2 (Cauchy transform)

For a Borel set $\Omega \subset \mathbb{C}$, we denote the *Cauchy transform* of Ω by $C^\Omega : \mathbb{C} \to \mathbb{C}$,

$$C^\Omega(w) = \frac{1}{\pi} \int_\Omega \frac{dA(\xi)}{w - \xi}$$

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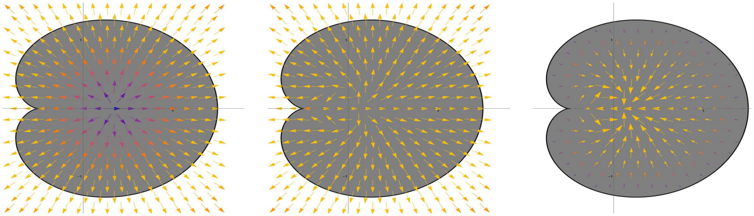

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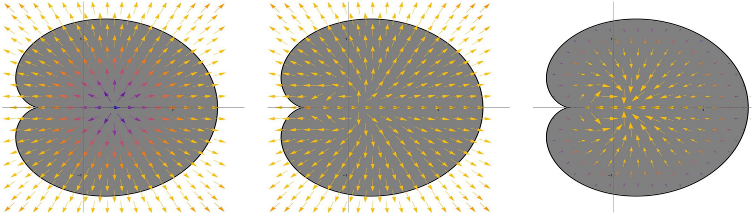

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$\mu = \overline{\partial} h$ corresponds to point charge distribution.

Schwarz function

Remark: $\Omega \subset \widehat{\mathbb{C}}$ is a QD iff it admits a *Schwarz function* $S : \Omega \to \widehat{\mathbb{C}}$.

[3] $\doteq$ denotes equality on the boundary.

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$$C^{\Omega^c}(w) = \lim_{r\to\infty} \frac{1}{\pi} \int_{\Omega^c \cap \mathbb{D}_r} \frac{dA(\xi)}{w - \xi}$$

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A $S-$function is a continuous map

$$S : \mathrm{Cl}(\Omega) \to \widehat{\mathbb{C}}$$

such that $S \in \mathcal{M}(\Omega)$ and[3]

$$S(w) \doteq \overline{w}$$

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Also,

$$S(w) = h(w) + C^{\Omega^c}(w), \quad w \in \Omega$$

(where C^{Ω^c} is understood in terms of its Cauchy principal value)[4]

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The Faber transform

Let $\Omega \subset \mathbb{C}$ be bounded and simply connected with Riemann map $\varphi : \mathbb{D} \to \Omega$,

$$\varphi(z) = f_0 + f_1 z + f_2 z^2 + \cdots$$

[5] $\mathcal{C}_A(X) =$ functions analytic in X, continuous up to ∂X, and $= 0$ at ∞.

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The associated *interior Faber transform* Φ_φ is a linear iso $\mathcal{C}_A(\mathbb{D}^c) \to \mathcal{C}_A(\Omega^c)$,[5]

$$\Phi_\varphi(f)(w) = \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \frac{f(z)\varphi'(z)}{\varphi(z) - w} dz = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f \circ \psi(\xi)}{\xi - w} d\xi$$

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Note: the Faber transform preserves polynomials and rational functions, e.g.

$$\Phi_\varphi\left(\frac{1}{z - z_0}\right)(w) = \frac{\varphi'(z_0)}{w - \varphi(z_0)}, \qquad F_n = \Phi_\varphi(z^n) \quad (n\text{th Faber polynomial})$$

Logarithms are also preserved.

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The Faber transform method

If $\Omega \in \mathrm{QD}(h)$ is s.c, with Riemann map φ, then φ **is rational** and[6]

$$\boxed{h = \Phi_\varphi\left(\varphi^\#\right)}$$

Chang & Makarov (2013)

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Example: $\Omega \in \mathrm{QD}\left(\sum_{j=1}^n \frac{c_j}{(w-w_0)^j}\right)$ is bounded (wlog assume $\varphi(0) = w_0$),

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$$\implies \sum_{j=1}^n \frac{c_j}{(w-w_0)^j} = \Phi_\varphi\left(\overline{w_0} + \sum_{j=1}^n \overline{\alpha_j} z^{-j}\right)(w) = \sum_{j=1}^n \frac{q_j(\boldsymbol{\alpha}, \overline{\boldsymbol{\alpha}})}{(w-w_0)^j}$$

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Definition 2.1 (Abelian quadrature domain)

We call a bounded domain $\Omega \subset \mathbb{C}$ an *Abelian quadrature domain* if there exists $h \in \widetilde{\text{Rat}}(\Omega)$,[7] ($h = r + L$ for some $r \in \text{Rat}(\Omega)$ and $e^L \in \text{Rat}(\Omega)$) such that

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Residue theorem:

$$\frac{1}{2\pi i} \oint_{\partial\Omega} f(w)h(w)dw = \sum_{k,j} c_{k,j} f^{(n_j)}(p_k) - \sum_j \alpha_j \int_{a_j}^{b_j} f(w)dw$$

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Remark: The Faber transform formula also applies to Abelian QDs.

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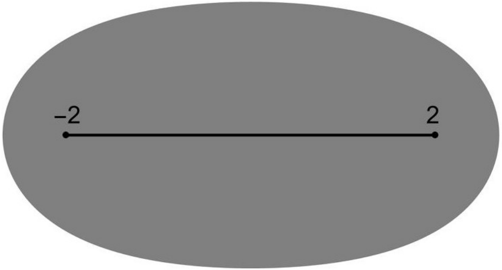
Abelian QD Example

The following ellipse-like region is an AQD:

$$\Omega = \left\{ w \in \mathbb{C} : \left| \tanh\left(\frac{w}{2}\right) \right|^2 < \tanh(1) \right\} \in \widetilde{\mathrm{QD}}\left(\ln\left(\frac{w+2}{w-2} \right) \right)$$



$$\frac{1}{\pi} \int_{\Omega} f dA = \frac{1}{2\pi i} \oint_{\partial\Omega} f(w) \ln\left(\frac{w+2}{w-2} \right) dw = \int_{-2}^{2} f(w) dw$$

Weighted quadrature domains

Definition 3.1 (Weighted quadrature domain)

We call a domain $\Omega \subset \widehat{\mathbb{C}}$ a *weighted quadrature domain* wrt the weight $\rho : \Omega \to \mathbb{R}_{\geq 0}$ if $\exists h \in \mathrm{Rat}(\Omega)$ s.t

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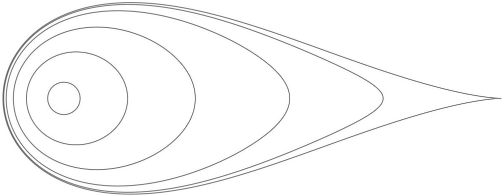
Example: if $\varphi(z) = aze^{az^{-1}}$ $(0 < a \leq 1)$, then $\Omega = \varphi(\mathbb{D}^-) \in \mathrm{QD}_{|w|^{-2}}(1)$



(Ω is unbounded component)

Weighted Faber transform formula

Recall that if $\Omega \in \mathrm{QD}(h)$ (bounded, s.c.), then $h = \Phi_\varphi(\varphi^\#)$.[8] This generalizes to certain classes of weighted QDs.

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If $\Omega \in \mathrm{QD}_{|R'|^2}\left(\frac{c}{w-w_0}\right)$ is bounded and s.c. with, $c > 0$, $w_0 \in \mathbb{C}$ "nice", then[9]

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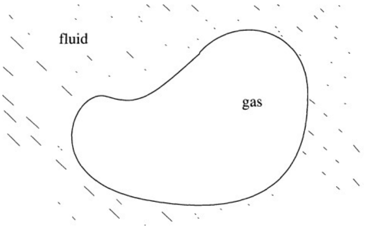

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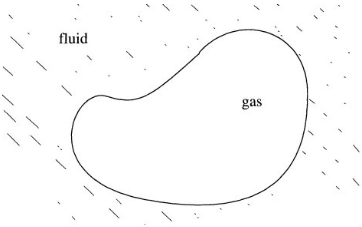

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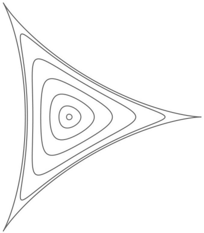

Deltoid

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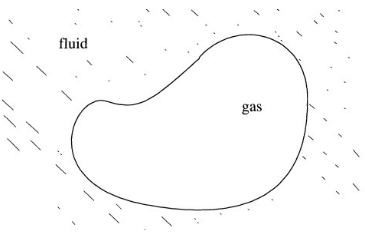

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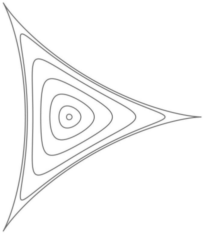

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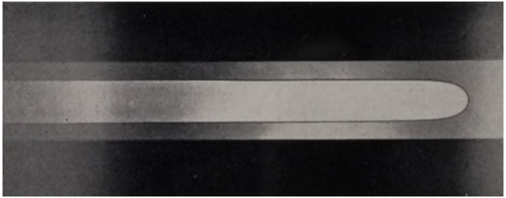
Remark: $\Omega_{t_0} \in \mathrm{QD}(h) \implies \Omega_{t_0+\delta t} \in \mathrm{QD}\left(h(w) + \frac{\delta t}{w-w_0}\right)$, where w_0 is the injection point.

Saffman-Taylor fingers



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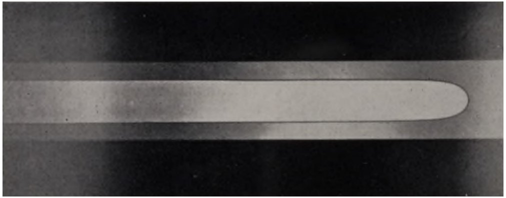

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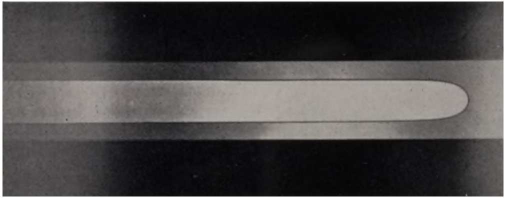

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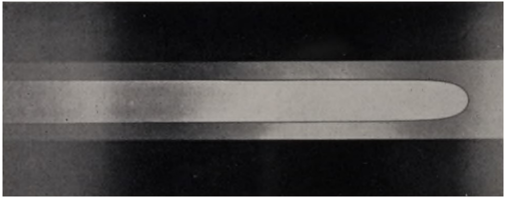

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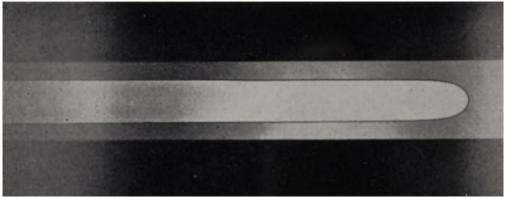

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Weighted Abelian QDs

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We call a bounded domain $\Omega \subset \mathbb{C}$ a weighted *Abelian quadrature domain* wrt the weights $\rho : \Omega \to \mathbb{R}_{\geq 0}$ and $\Lambda \in \mathrm{Rat}(\Omega^c)$ if $\exists h = r + \Lambda L$ for $r, e^L \in \mathrm{Rat}(\Omega)$, such that

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Unbounded case is similar.

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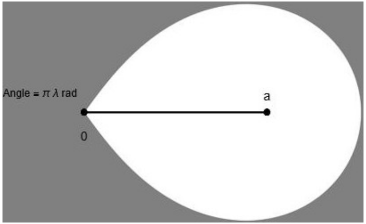

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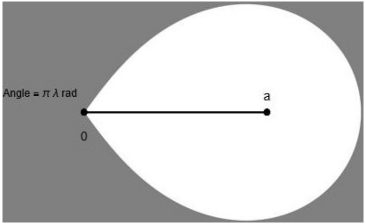

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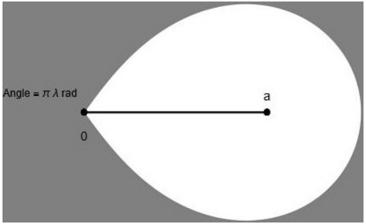

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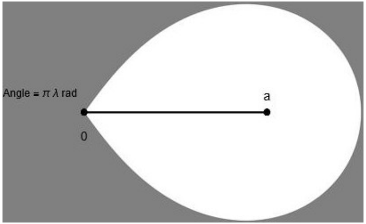

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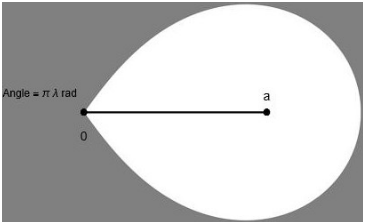

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So if $f \in L^1_a(a\mathbb{D}(1)^{\lambda}; |w|^{-2})$,

$$\frac{1}{\pi}\int_{a\mathbb{D}(1)^{\lambda}} \frac{f(w)}{|w|^2}\,dA(w) = \lambda\int_0^a \frac{f(w)}{w}\,dw$$

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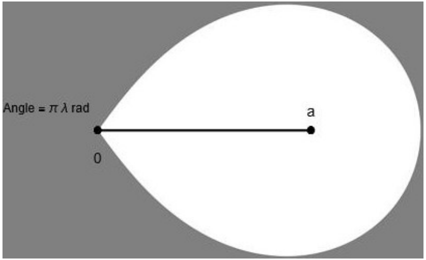
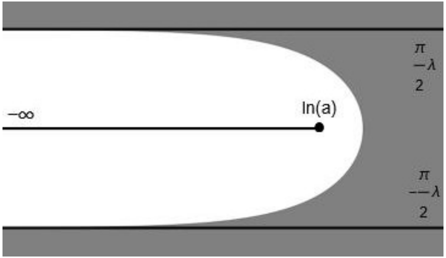
Weighted Abelian QD Example (cont.)

Changing variables $w \mapsto \ln(w)$: (Saffman-Taylor finger)



$a\mathbb{D}(1)^{\lambda}$

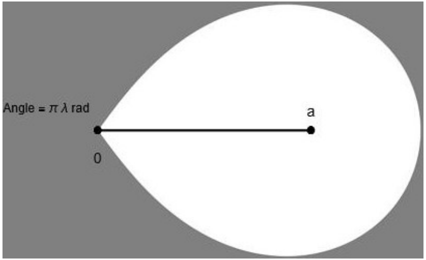
$$\xrightarrow[\frac{dA}{|w|^2} \mapsto dA]{w \mapsto \ln(w)}$$



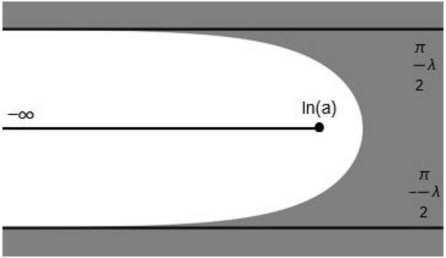
$\ln(a) + \lambda \ln(\mathbb{D}(1))$

Weighted Abelian QD Example (cont.)

Changing variables $w \mapsto \ln(w)$: (Saffman-Taylor finger)



$$\xrightarrow[\frac{dA}{|w|^2} \mapsto dA]{w \mapsto \ln(w)}$$



$a\mathbb{D}(1)^{\lambda}$ $\qquad$ $\ln(a) + \lambda \ln(\mathbb{D}(1))$

Gives

$$\frac{1}{\pi}\int_{\ln(a)+\lambda\ln(\mathbb{D})} f dA = \lambda \int_{-\infty}^{\ln(a)} f(w)dw$$

For $f \in L^1_a(\ln(a) + \lambda \ln(\mathbb{D}(1)))$

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That's all!

Thank you!

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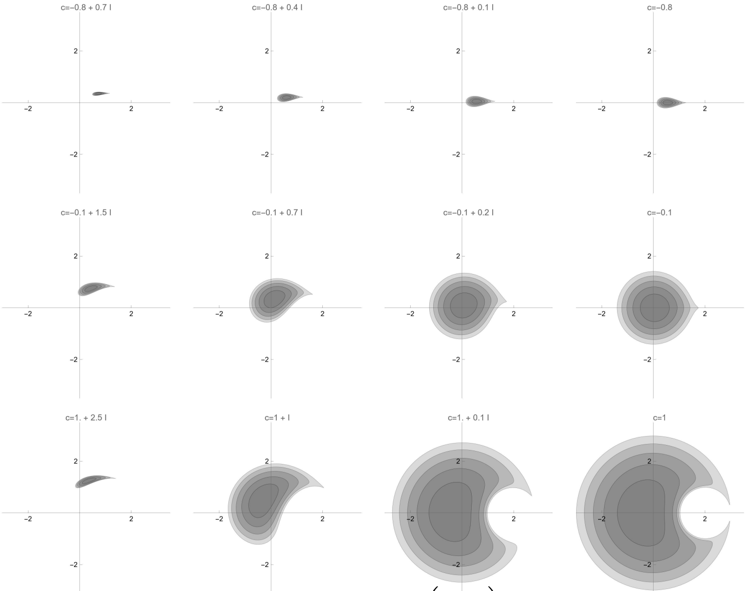
$$\varphi(z) = az\frac{z - z_1}{z - \overline{z_0}^{-1}}, \qquad (\varphi(z_0) = w_0)$$

One point UQDs



Antiholomorphic dynamics

If Ω is a QD, can consider dynamics of Schwarz reflection, $\{\sigma^{\circ n}\}$, $\sigma = \overline{S}$.

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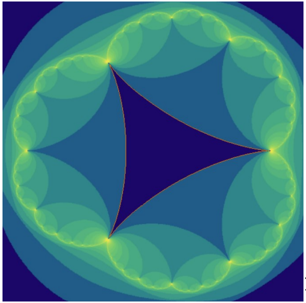
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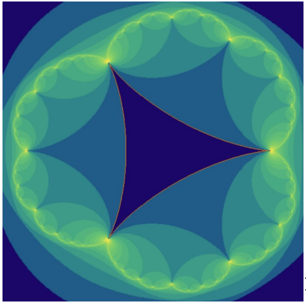[13]

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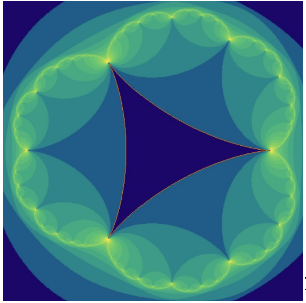
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